ALL SAINTS' COLLEGE

Mathematics Specialist

Test 4 2016

Integration Techniques & Applications of Integral Calculus

NAME: Solutions

TEACHER: MLA

Resource Free Section

30 marks
30 minutes

Question 1 [~~2~~ 3, ~~3~~ 3, 3 & 3 = ~~11~~ 9 marks]

Determine the following indefinite integrals:

(a) ~~$\int (x^2 - 1)\, 12\cos(3x - x^3)\, dx$~~

(b) $\int 5\tan^2(5x)\,dx$

Note. $1 + \tan^2(x) = \sec^2(x)$

$= \int 5\,[\sec^2(5x) - 1]\,dx$ ✓

Form : $\int f'(x)\,.\,\sec^2 f(x)\,dx$

$= \int 5\sec^2(5x)\,dx - 5\int 1\,dx$

$= \tan f(x) + c$

$= \tan(5x) + c_1 - 5x + c_2$

$= \tan(5x) - 5x + K$ //

(c) $\int 27\tan^2(3x)\sec^2(3x)\,dx$

$= 9\int 3\sec^2(3x)\,.\,\tan^2(3x)\,dx$ //

Form: $\int f'(x)\,.\,[f(x)]^n\,dx$

where $f(x) = \tan(3x)$

$= 9\,.\,\dfrac{\tan^3(3x)}{3} + c$

$= 3\tan^3(3x) + c$ ✓

(d) $\int 8\sin^2(2x)\,dx$

Note. $2\sin^2(2x) = 1 - \cos(4x)$

$= 4\int 2\sin^2(2x)\,dx$ ✓

$= 4\int [1 - \cos(4x)]\,dx$ ✓

$= \int 4\,dx - \int 4\cos(4x)\,dx$

Form; $\int f'(x)\,.\,\cos f(x)\,dx$

$= 4x + c_1 - \sin(4x) + c_2$

$= \sin f(x) + c$

$= 4x - \sin(4x) + K$ ✓

9.

Question 2 **[~~2 & 3~~ = 5 marks]**

(a) Use the substitution $u = 1 + 2x$ to determine the indefinite integral $\int \frac{x}{1+2x}\, dx$

$$\int \frac{x}{1+2x}\, dx$$

$u = 1 + 2x \Rightarrow \frac{u-1}{2} = x$

$\frac{du}{dx} = 2$

$$= \int \frac{x}{u} \cdot \frac{dx}{du} \cdot du$$

$$= \int \frac{u-1}{2u} \cdot \frac{1}{2} \cdot du$$

$$= \int \frac{u-1}{4u}\, du$$

$$= \int \frac{u}{4u}\, du - \int \frac{1}{4u}\, du \quad ✓ \qquad \text{or} \rightarrow \int \frac{1}{4}\, du - \frac{1}{4}\int \frac{1}{u}\, du$$

$$= \int \frac{1}{4}\, du - \frac{1}{4}\int \frac{4}{4u}\, du \qquad = \frac{u}{4} + c_1 - \frac{1}{4}\ln|u| + c_2$$

$$= \frac{u}{4} + c_1 - \frac{1}{4}\ln|4u| + c_2 \qquad = \frac{1+2x - \ln|1+2x|}{4} + k$$

$$= \frac{1+2x - \ln|4+8x|}{4} + k \quad ✓$$

(b) Use the substitution $u = 1 + \sin(x)$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{4\cos(x)}{\sqrt{1+\sin(x)}}$

$$\int_0^{\frac{\pi}{2}} \frac{4\cos(x)}{\sqrt{1+\sin(x)}}\, dx$$

$u = 1 + \sin(x)$

$\frac{du}{dx} = \cos(x)$

$$= \int_1^2 \frac{4\cos(x)}{\sqrt{u}} \cdot \frac{1}{\cos(x)} \cdot du \quad ✓✓$$

when $x = 0$, $u = 1$

when $x = \frac{\pi}{2}$, $u = 2$ ✓

$$= \int_1^2 4u^{-\frac{1}{2}}\, du \quad ✓$$

$$= \left[8\sqrt{u}\right]_1^2$$

$$= 8\sqrt{2} - 8 \quad ✓$$

5.

Question 3 **[3 & 3 = 6 marks]**

(a) If $f'(x) = \cos(x)\sin(2x)$, determine $f(x)$.

$$F(x) = \int \cos(x)\sin(2x)\,dx$$

$$= \int \cos(x).\,2\sin(x).\cos(x)\,dx \quad \checkmark$$

$$= \int 2\sin(x).\cos^2(x)\,dx$$

$$= -2\int -\sin(x).\cos^2(x)\,dx \quad \checkmark$$

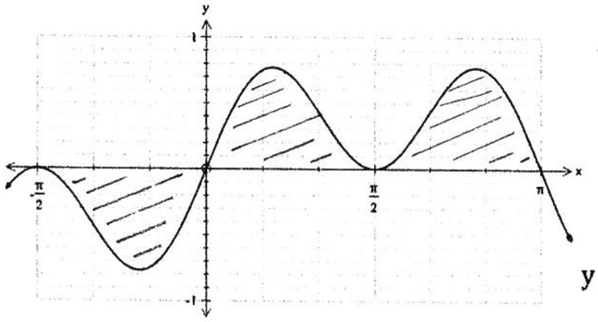
$$= -2.\frac{\cos^3(x)}{3} + c \quad \checkmark$$

Form: $\int f'(x).[f(x)]^n\,dx$

$$= \frac{[f(x)]^{n+1}}{n+1} + c$$

where $f(x) = \cos(x)$

(b) Hence, calculate the area between the curve $y = \cos(x)\sin(2x)$ and the x-axis from $x = -\frac{\pi}{2}$ to $x = \pi$.



$y = \cos(x)\sin(2x)$

$$\text{Area} = 3\left[\frac{-2\cos^3(x)}{3}\right]_0^{\frac{\pi}{2}} \quad \checkmark\checkmark$$

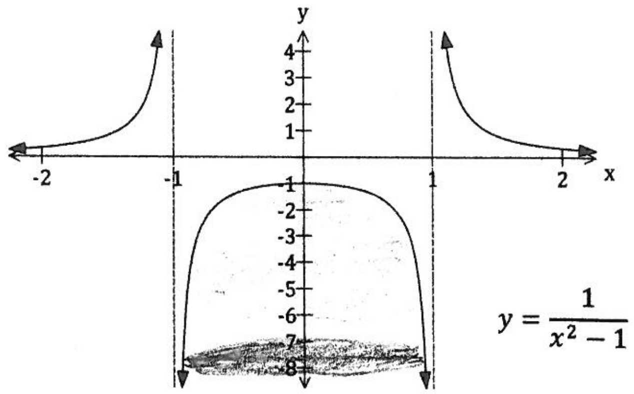
$$= -2\cos^3\left(\frac{\pi}{2}\right) + 2\cos^3(0)$$

$$= 2 \text{ units}^2$$

6

Question 4 **[~~4~~ 5 marks]**

Calculate the exact volume generated by revolving the area trapped between $y = \frac{1}{x^2-1}$, the vertical axis and the lines $y = -e^2$ and $y = -1$ about the y axis.



$$y = \frac{1}{x^2-1}$$

$$x^2 - 1 = \frac{1}{y}$$

$$x^2 = \frac{1}{y} + 1$$

$$x = \pm\sqrt{\frac{1}{y}+1} = f(y)$$

$$V = \pi \int [f(y)]^2 \, dy$$

$$V = \pi \int_{-e^2}^{-1} \left(\sqrt{\frac{1}{y}+1}\right)^2 dy \quad //$$

$$= \pi \int_{-e^2}^{-1} \left(\frac{1}{y}+1\right) dy$$

$$= \pi \Big[y + \ln|y| \Big]_{-e^2}^{-1} \quad ✓$$

$$= \pi \Big[\left(-1 + \ln|-1|\right) - \left(-e^2 + \ln|-e^2|\right) \Big] \quad ✓$$

$$= \pi \left[-1 + \ln(1) + e^2 - \ln(e^2) \right]$$

$$= \pi \left[-1 + 0 + e^2 - 2\ln(e) \right] \qquad \text{Note: } \ln(e) = 1$$

$$= \pi \left(-1 + e^2 - 2 \right)$$

$$= \pi \left(e^2 - 3 \right) \quad ✓$$

5.

3 5

Question 5 [2 & ~~2~~ = ~~4~~ marks]

(a) If $y = \ln(x^{x^2})$, determine $\frac{dy}{dx}$

Hint 1: Apply a suitable log law to $y = \ln(x^{x^2})$ before differentiating

Hint 2: Do not factorise your final answer

$y = x^2 . \ln|x|$ ✓ ... log law applied to $\ln(x)^{x^2}$

$\frac{dy}{dx} = \ln|x| . 2x + x^2 \frac{1}{x}$ ... product rule

$= 2x . \ln|x| + x$ ✓

or: $\int 2x \ln|x| \, dx = \int (2x \ln|x| + x - x) \, dx$

$= \int (2x \ln|x| + x) \, dx - \int x \, dx$

$= x^2 \ln|x| - \frac{x^2}{2} + c$

(b) Hence, find $\int 2x \ln(x) \, dx$

$x^2 \ln|x| = \int (2x \ln|x| + x) \, dx$ ... from (a) ✓

$= \int 2x \ln|x| \, dx + \int x \, dx$

$x^2 \ln|x| - \int x \, dx = \int 2x \ln|x| \, dx$ ✓

$\therefore x^2 \ln|x| - \frac{x^2}{2} + c = \int 2x \ln|x| \, dx$ ✓

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Resource Rich Section

20 marks
20 minutes

One unfolded A4 page of notes, SCSA formulae booklet and ClassPad calculator permitted

1 2

Question 6 [~~2~~ & 1 = ~~3~~ marks]

(a) Express $\int \frac{x^2-x+1}{(x+3)(x^2+4)}\,dx$, in exact terms

ClassPad : $\ln|x+3| - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + c$ ✓

(b) Evaluate $\int_0^{4\pi} \frac{x^2-x+1}{(x+3)(x^2+4)}\,dx$, correct to 2 decimal places

0.94 ✓

6

Question 7 [~~5~~ marks]

Use your knowledge of partial fractions to determine $\int \frac{7x^2-2x+5}{(x-1)(x^2+1)}\,dx$

Show clear working.

$$\frac{7x^2-2x+5}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \quad ✓$$

$$= \frac{A(x^2+1)+(Bx+C)(x-1)}{(x-1)(x^2+1)}$$

$$= \frac{Ax^2+A+Bx^2-Bx+Cx-C}{(x-1)(x^2+1)} \quad ✓$$

Equating coefficients : $\left.\begin{array}{l} A+B=7 \\ C-B=-2 \\ A-C=5 \end{array}\right\}$ Solving simultaneously on Classpad : $\left.\begin{array}{l} A=5 \\ B=2 \\ C=0 \end{array}\right\}$ ✓✓

So, $\int \frac{7x^2-2x+5}{(x-1)(x^2+1)}\,dx = \int \frac{5}{x-1}\,dx + \int \frac{2x}{x^2+1}\,dx$

$= 5\ln|x-1| + c_1 + \ln|x^2+1| + c_2$

$= 5\ln|x-1| + \ln|x^2+1| + K$

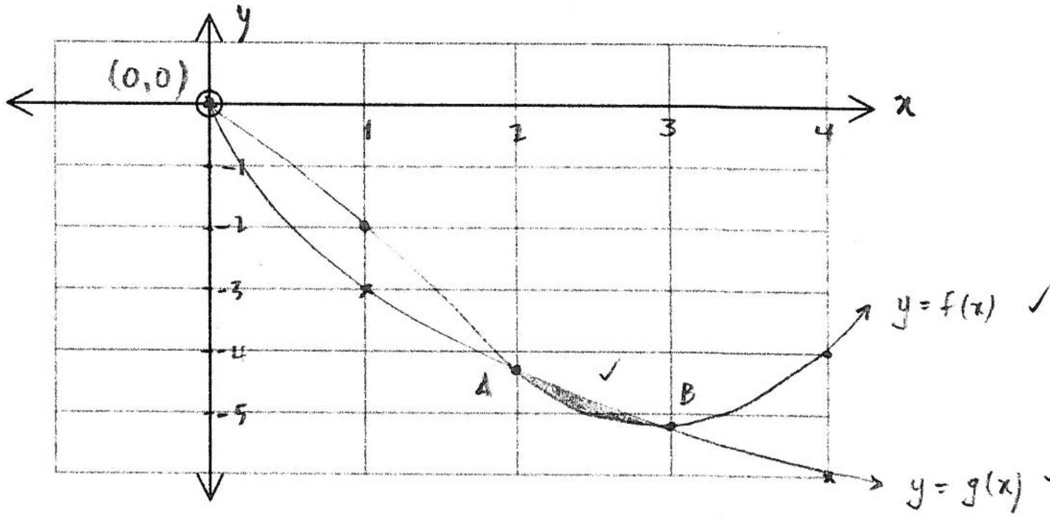
✓ ✓

Question 8 [3, 2, 2 & 1 = 8 marks]

Consider the functions $f(x) = \frac{\sqrt{x}\,(x^2-5x)}{2}$ and $g(x) = -3\sqrt{x}$

A, B and (0, 0) are the three points of intersection of the aforementioned functions.

(a) Draw a neat sketch of f(x) and g(x) on the axes below. Label points A and B.



(b) State the ordered pairs for the points A and B, correct to 2 decimal places.

A : (2, −4·24) ; B = (3, −5.20)

✓ ✓

(c) Write an expression for the area enclosed by the graphs of f(x) and g(x).

Area enclosed = $\int_2^3 [g(x) - f(x)]\,dx$ ✓✓

(d) Use your Classpad to determine the area trapped between f and g

0.1316 units^2 (4 dp) ✓

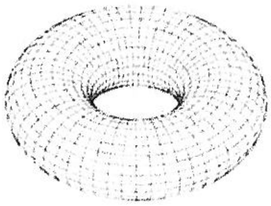
note. use : Analysis → G-Solve → Integral → ∫dx Intersection,

with lower bound = 2, upper bound = 3.

8.

Question 9 **[4 marks]**

In geometry, a torus is a surface of revolution generated by revolving a circle in 3-dimensional space about an axis co-planar with the circle.



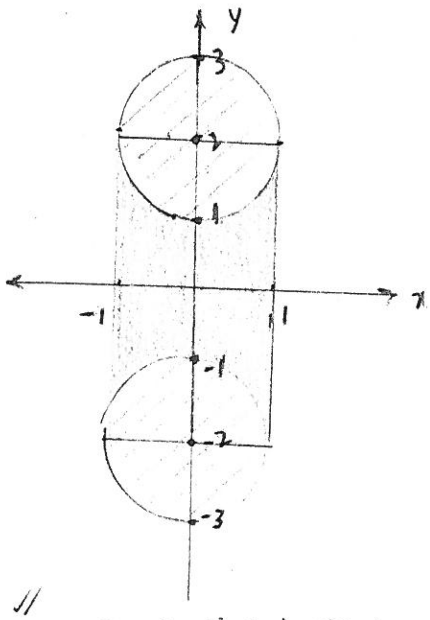
Use calculus to determine the volume of the torus formed by rotating the circle with equation $x^2 + (y-2)^2 = 1$ about the x-axis.

Circle : centre $(0,2)$; radius $= 1$

$(y-2)^2 = 1-x^2$

$y-2 = \pm\sqrt{1-x^2}$

$y = 2 \pm \sqrt{1-x^2}$ ✓



x- sectional view

Volume of Torus

= volume obtained by rotating area beneath top half of circle $(2+\sqrt{1-x^2})$

subtract

volume obtained by rotating area beneath bottom half of circle $(2-\sqrt{1-x^2})$

$= \pi \int_{-1}^{1} \left(2+\sqrt{1-x^2}\right)^2 dx - \pi \int_{-1}^{1} \left(2-\sqrt{1-x^2}\right)^2 dx$ //

$= 39.478$ units^3 ✓ or $4\pi^2$ units^3

note. Volumes $= 49.0607 - 9.5823$

Note. Could use Analysis → G-Solve → $\pi\int f(x)^2 dx$ for top and bottom halves of the circle in Graphs + Tables.

4.